

An effective bound for the gonality conjecture

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1. INTRODUCTION

Ein and Lazarsfeld have shown that one can read off the gonality of an algebraic curve from its syzygies in the embedding defined by any one line bundle of sufficiently large degree. This note extends their approach and shows that the gonality can be detected from the syzygies of an embedding by any line bundle of degree at least $4g - 3$.

Let $C \subset \mathbf{P}H^0(L) = \mathbf{P}^r$ be a smooth complex projective curve of genus $g \geq 2$, embedded by a very ample line bundle L of degree d .

We denote by $S = \text{Sym } H^0(C, L)$ the homogeneous coordinate ring of \mathbf{P}^r , by $R = R(L) = \bigoplus_m H^0(C, mL)$ the graded S -module associated to L , and by $E_\bullet = E_\bullet(L)$ the minimal graded free resolution of R over S :

$$0 \longrightarrow E_{r-1} \longrightarrow \dots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \longrightarrow R \longrightarrow 0.$$

Let $K_{p,q}(C; L)$ be the vector space of minimal generators of E_p in degree $p + q$, so that

$$E_p = \bigoplus_q K_{p,q}(C; L) \otimes S(-p - q).$$

If L is normally generated, then $E_0 = S$, and the remainder of the complex E_\bullet is a minimal graded resolution of the homogeneous ideal I_{C/\mathbf{P}^r} of C in \mathbf{P}^r .

If L is nonspecial then $K_{p,q}(C; L)$ vanishes for $q \geq 3$, hence for $\deg(L) \geq 2g + 1$ the resolution essentially consists of two strands $K_{p,1}(C; L)$ and $K_{p,2}(C; L)$.

For a discussion of these strands, see [8] and [6, ch. 8B].

The gonality conjecture of Green and Lazarsfeld [8] concerns the “quadratic strand” $K_{p,1}(C; L)$ for line bundles of large degree. If C has a basepointfree pencil of degree k , then the linear spans of the fibers of this pencil form a k -dimensional rational normal scroll S . The minimal graded resolution of its ideal I_{S/\mathbf{P}^r} has a linear strand of length $r - k$ which embeds into the resolution of I_{C/\mathbf{P}^r} .

Green and Lazarsfeld conjectured in 1986 and Ein and Lazarsfeld proved in 2014 that such pencils determine the range of values for p where $K_{p,1}(C; L)$ does not vanish, if the degree of L is large.

We show that this already holds for $\deg L \geq 4g - 3$.

Theorem 1.1. *If $H^1(L \otimes K^{-1}) = 0$, then*

$$K_{p,1}(C; L) \neq 0 \iff 1 \leq p \leq r - \text{gon}(C).$$

In particular, the equivalence holds if $\deg(L) \geq 4g - 3$.

Although our result represents a significant step forward, we do not expect the bound $4g - 3$ to be optimal. Aprodu and Voisin [1], [3] found that $\deg(L) \geq 3g$ suffices for a general curve of each gonality. Green [7] had earlier demonstrated the conclusion for $p = r - 1$ resp. $p = r - 2$ under even lower bounds of $\deg(L) \geq 2g + 1$ resp. $\deg(L) \geq 2g + 2$ and $r \neq 5$ (the exceptions for $r = 5$ are smooth curves of genus 3 and degree $8 = 2g + 2$ on the Veronese surface in \mathbf{P}^5).

While it is not hard to exhibit a non-vanishing syzygy from a pencil on C , it is not known how to construct a pencil out of a non-vanishing syzygy, except for the cases $p = r - 1$ and $p = r - 2$ covered by Green. Ein und Lazarsfeld's proof works by contradiction. They assume that C does not have a pencil of low degree, and prove the vanishing of the related syzygy module.

Following their arguments, 1.1 can be derived from a more general result on the syzygies associated to a line bundle B . Letting $R = R(B; L) = \oplus_m H^0(C, B + mL)$, its minimal graded free resolution $E_\bullet(B; L)$ over S gives rise to Koszul cohomology groups $K_{p,q}(C, B; L)$.

Recall that B is called p -very ample if every effective divisor ξ of degree $(p + 1)$ on C imposes independent conditions on the sections of B ; e.g., the canonical divisor K_C is p -very ample if and only if C does not have a pencil of degree $< p + 2$.

Our main result is a strengthening of one direction of [5, Thm. B].

Theorem 1.2. *Let B be a p -very ample line bundle. If $H^1 L = 0 = H^1(B^{-1} \otimes L)$, then $K_{p,1}(C, B; L) = 0$.*

As explained by Ein and Lazarsfeld, Theorem 1.1 follows from setting $B = K_C$ in 1.2, using the duality between $K_{p,q}(C, B; L)$ and $K_{r-1-p,2-q}(C, K_C \otimes B^{-1}; L)$.

For the proof of 1.2, we use the representation of $K_{p,1}(C, B; L)$ as cokernel of a map of global sections of sheaves on the symmetric product C_{p+1} [5].

Our analysis starts by transferring the question from the symmetric to the cartesian product. Denoting by $pr_{p+2}: C^{p+2} \rightarrow C$ resp. $\pi_{p+2}: C^{p+2} \rightarrow C^{p+1}$ the projection maps on the $(p + 2)$ -nd resp. first $p + 1$ components, by $\Delta_{i,j}$ the diagonals $x_i = x_j$ in C^{p+2} , we set

$$M_B = M_{p+1,B} = \pi_{p+2,*}(pr_{p+2}^*(B) \otimes \mathcal{O}(-\sum_{i=1}^{p+1} \Delta_{i,p+2})),$$

and we show that if B is p -very ample, the vanishing of $K_{p,1}(C, B; L)$ follows from the vanishing of

$$H^1(C^{p+1}, M_B \otimes (\otimes_{i=1}^{p+1} pr_i^*(L)) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})).$$

There are two standard approaches for proving such vanishing theorems:

- (i) using a filtration of M_B , and

(ii) using a resolution of M_B derived from a representation of M_B as a kernel bundle.

If we have no information on B except that it is p -very ample, only method (ii) is at our disposal. Theorem 1.2 represents this case (proved in 3.1), and the required representation of M_B as a kernel bundle is given by

$$0 \longrightarrow M_{p+1,B} \longrightarrow \pi_{p+1}^* M_{p,B} \longrightarrow pr_{p+1}^*(B) \otimes \mathcal{O}\left(-\sum_{i=1}^p \Delta_{i,p+1}\right) \longrightarrow 0$$

which allows us to proceed by induction.

If there is a point $x \in C$ such that $B(-x)$ is again p -very ample, then we can start to filter M_B as

$$0 \longrightarrow M_{B(-x)} \longrightarrow M_B \longrightarrow \mathcal{O}(-x, \dots, -x) \longrightarrow 0.$$

Improved bounds can be obtained if the degree of B is large compared to g (see 3.2 and 3.3); for $B = K_C$ (the canonical bundle) this approach leads to an improvement if and only if K is actually $(p+1)$ -very ample (see 4.3).

What happens if K is not $(p+1)$ -very ample? For $p = 0$, i.e., hyperelliptic C , the bundle $M_{1,B}$ is the direct sum of $g-1$ copies of $(g_2^1)^{-1}$, and $K_{0,1}(C, K; L)$ vanishes for $\deg L \geq 2g+1$. For $p = 1$ we know from Green's work that $\deg L \geq 2g+2$ ($r \neq 5$) suffices. It seems likely that similar improvements are possible for higher p ; however, they require a different approach.

In section 2 we recall Voisin's representation of the Koszul cohomology groups, following Ein and Lazarsfeld. The third section contains the proofs of our main results. Section 4 discusses improvements for $B = K$, taking the geometry of the canonical embedding into account.

2. REVIEW OF THE SETUP

We recall the setup used by Ein and Lazarsfeld.

$C \subset \mathbf{P}H^0(L) = \mathbf{P}^r$ is a smooth complex projective curve of genus g , embedded by a very ample line bundle L of degree d . Given another line bundle B on C , let

$$K_{p,q}(B; L) = K_{p,q}(C, B; L)$$

be the cohomology groups of the complex

$$\begin{aligned} \wedge^{p+1} H^0(L) \otimes H^0(B \otimes L^{\otimes(q-1)}) &\longrightarrow \wedge^p H^0(L) \otimes H^0(B \otimes L^{\otimes q}) \\ &\longrightarrow \wedge^{p-1} H^0(L) \otimes H^0(B \otimes L^{\otimes(q+1)}). \end{aligned}$$

Denoting by C_k the k -th symmetric product of C , the map

$$\sigma_{p+1}: C \times C_p \longrightarrow C_{p+1}, \quad (x, \xi) \mapsto x + \xi$$

realizes $C \times C_p$ as the universal family of degree $p+1$ divisors over C_{p+1} .

Now let

$$E_B = E_{p+1,B} = \sigma_{p+1,*} pr_1^*(B)$$

where pr_i is the projection on the i -th factor. E_B is a vector bundle of rank $p+1$ on C_{p+1} , and $H^0(C_{p+1}, E_B) = H^0(C, B)$. We therefore have a homomorphism

$$(1) \quad \text{ev}_B = \text{ev}_{p+1,B}: H^0(C, B) \otimes \mathcal{O}_{C_{p+1}} \longrightarrow E_B$$

of vector bundles on C_{p+1} . By definition, ev_B is surjective if and only if B is p -very ample. Setting

$$N_L = N_{p+1,L} = \det E_L,$$

$\wedge^{p+1} \text{ev}_L$ determines an isomorphism [11]

$$\wedge^{p+1} H^0(C, L) \longrightarrow H^0(C_{p+1}, N_L).$$

Tensoring ev_B by N_L , now consider the following map of vector bundles on C_{p+1} :

$$(2) \quad H^0(C, B) \otimes N_L \longrightarrow E_B \otimes N_L :$$

Lemma 2.1 (Voisin, Ein-Lazarsfeld [5, Lemma 1.1]). *The global sections of $E_B \otimes N_L$ can be identified with the space*

$$Z_{p,1}(B; L) = \text{Ker} \left(\wedge^p H^0(L) \otimes H^0(B \otimes L) \longrightarrow \wedge^{p-1} H^0(L) \otimes H^0(B \otimes L^{\otimes 2}) \right)$$

of Koszul cycles. Under this identification, the homomorphism

$$H^0(C, B) \otimes H^0(C_{p+1}, N_L) = H^0(C, B) \otimes \wedge^{p+1} H^0(C, L) \longrightarrow H^0(C_{p+1}, E_B \otimes N_L)$$

arising from (2) agrees with the Koszul differential. In particular,

$$K_{p,1}(C, B; L) = 0$$

if and only if the bundle map (2) is surjective on global sections.

Ein and Lazarsfeld complete their proof by applying general vanishing theorems to the kernel bundle of (2). They obtain a bound of $\deg(L) > (p^2 + p + 2)(g - 1) + (p + 1)\deg(B)$ as a sufficient condition for the vanishing.

Our contribution starts at this point. We begin by transferring the question from the symmetric product to the cartesian product.

Denote by $\pi = \pi_n: C^n \rightarrow C^{n-1}$ the projection on the first $n-1$ components (forget n -th component) and by $r = r_n: C^n \rightarrow C_n$ the canonical projection.

The map (2) pulls back by r , and by taking global sections, we obtain a commutative diagram

$$(3) \quad \begin{array}{ccc} H^0(C, B) \otimes H^0(N_L) & \xrightarrow{H^0(\text{ev}_B \otimes N_L)} & H^0(E_B \otimes N_L) \\ \downarrow & & \downarrow \\ H^0(C, B) \otimes H^0(r^* N_L) & \xrightarrow{H^0(r^*(\text{ev}_B \otimes N_L))} & H^0(r^*(E_B \otimes N_L)). \end{array}$$

Lemma 2.2. *If the horizontal map on the bottom of the diagram (3) is surjective, then the horizontal map on the top is also surjective.*

Proof. Given a section s_0 in the top right, consider its image s in the bottom right cohomology group. This image is invariant under the action of the symmetric group S_{p+1} . Provided that the bottom horizontal map is surjective, we can average a preimage s' in the bottom left over all translates (using the characteristic 0 assumption) in order to arrive at an S_{p+1} -invariant preimage s'' . The S_{p+1} -invariance then implies [2, (5.2)] that s'' lies in the image of the (injective) vertical map from the top left. This provides the sought after preimage of s_0 . \square

3. VANISHING RESULTS

We continue in the setup from section 2:

$C \subset \mathbf{P}H^0(L) = \mathbf{P}^r$ is a smooth complex projective curve of genus g , embedded by a very ample line bundle L of degree d , B a p -very ample line bundle on C . We work on the cartesian product C^{p+2} .

The map ev_B from (1) above pulls back to a map of sheaves on C^{p+1}

$$\text{ev}' = r^*(\text{ev}_B): H^0(C, B) \otimes \mathcal{O}_{C^{p+1}} \longrightarrow r^*E_B$$

that arises from the canonical map

$$\mathcal{O}_{C^{p+2}} \longrightarrow \mathcal{O}_{\sum_{i=1}^{p+1} \Delta_{i,p+2}}$$

by tensoring with $pr_{p+2}^*(B)$ and applying $\pi_{p+2,*}$. Further we have

$$r^*N_L = \otimes_{i=1}^{p+1} pr_i^*(L) \otimes \mathcal{O}\left(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j}\right) \quad [2, \text{proof of (5.2)}].$$

Denoting the vector bundle

$$M_B = M_{p+1,B} = \pi_{p+2,*}(pr_{p+2}^*(B) \otimes \mathcal{O}(-\sum_{i=1}^{p+1} \Delta_{i,p+2})),$$

the surjectivity of the bottom horizontal map of diagram (3) from the previous section would follow from

$$(4) \quad H^1(C^{p+1}, M_B \otimes (\otimes_{i=1}^{p+1} pr_i^*(L)) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0$$

We now show by induction on p more generally

Theorem 3.1. *If L is non-special and $H^1(C, B^{-1} \otimes L) = 0$, then we have*

$$(5) \quad H^k(C^{p+1}, \wedge^m M_B \otimes pr_1^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0$$

for all $k > 0$, $m > 0$.

Proof. First consider $p = 0$: In this case we know that B is globally generated and we have an exact sequence

$$0 \longrightarrow M_{1,B} \longrightarrow H^0 B \otimes \mathcal{O}_C \longrightarrow B \longrightarrow 0.$$

Replacing $H^0 B$ by a general subspace of dimension 2 to define a line bundle $M'_{1,B}$ sitting in an exact sequence

$$0 \longrightarrow M'_{1,B} \longrightarrow M_{1,B} \longrightarrow \oplus \mathcal{O}_C \longrightarrow 0,$$

we note that $M'_{1,B} = B^{-1}$, hence $M_{1,B}$ and all its exterior powers have a filtration whose components are isomorphic to either $M'_{1,B}$ or \mathcal{O}_C . The vanishing now follows from the assumptions of $H^1(B^{-1} \otimes L) = 0 = H^1 L$.

For the induction step we need to consider several cases:

Case 1. $\text{rank}(M_B) = m$: The exact sequence on C^{p+2}

$$0 \longrightarrow \mathcal{O}\left(-\sum_{i=1}^{p+1} \Delta_{i,p+2}\right) \longrightarrow \mathcal{O}\left(-\sum_{i=1}^p \Delta_{i,p+2}\right) \longrightarrow \mathcal{O}\left(-\sum_{i=1}^p \Delta_{i,p+2}\right) \otimes \mathcal{O}_{\Delta_{p+1,p+2}} \longrightarrow 0$$

yields, after tensoring with $pr_{p+2}^*(B)$ and applying $\pi_{p+2,*}$, an exact sequence on C^{p+1}

$$(6) \quad 0 \longrightarrow M_{p+1,B} \longrightarrow \pi_{p+1}^*(M_{p,B}) \longrightarrow pr_{p+1}^*(B) \otimes \mathcal{O}\left(-\sum_{i=1}^p \Delta_{i,p+1}\right) \longrightarrow 0.$$

We conclude inductively that

$$\begin{aligned} \det M_{p+1,B} &= \det(\pi_{p+1}^*(M_{p,B})) \otimes pr_{p+1}^*(B^{-1}) \otimes \mathcal{O}\left(\sum_{i=1}^p \Delta_{i,p+1}\right) \\ &= pr_1^*(B^{-1}) \otimes \cdots \otimes pr_{p+1}^*(B^{-1}) \otimes \mathcal{O}\left(\sum_{1 \leq i < j \leq p+1} \Delta_{i,j}\right), \end{aligned}$$

and find

$$\begin{aligned} H^k(C^{p+1}, \det(M_B) \otimes pr_1^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}\left(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j}\right)) \\ = H^k(C^{p+1}, pr_1^*(L \otimes B^{-1}) \otimes \cdots \otimes pr_{p+1}^*(L \otimes B^{-1})) = 0 \end{aligned}$$

by Künneth for $k > 0$.

Case 2. $\text{rank}(M_B) > m$: The exterior powers of the sequence (6) lead to a resolution of $\wedge^m M_{p+1,B}$ which looks as follows

$$\begin{aligned} \cdots \longrightarrow \wedge^{m+2} \pi_{p+1}^*(M_{p,B}) \otimes pr_{p+1}^*(B^{-2}) \otimes \mathcal{O}\left(\sum_{i=1}^p \Delta_{i,p+1}\right)^{\otimes 2} \\ \longrightarrow \wedge^{m+1} \pi_{p+1}^*(M_{p,B}) \otimes pr_{p+1}^*(B^{-1}) \otimes \mathcal{O}\left(\sum_{i=1}^p \Delta_{i,p+1}\right) \longrightarrow \wedge^m M_{p+1,B} \longrightarrow 0. \end{aligned}$$

The desired vanishing (5) will result from the vanishings of

$$(7) \quad H^{k+l}(C^{p+1}, \wedge^{m+l+1} \pi_{p+1}^*(M_{p,B}) \otimes pr_1^*(L) \otimes \cdots \otimes pr_p^*(L) \otimes pr_{p+1}^*(L \otimes B^{-l-1}) \\ \otimes \mathcal{O}(\sum_{i=1}^p \Delta_{i,p+1})^{\otimes(l+1)} \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j}))$$

for all $l \geq 0$ (see e.g., [10, B.1.2 (i)]).

Case 2.1. $l = 0$: The diagonals involving the component $p + 1$ disappear. We need to consider

$$H^k(C^{p+1}, \wedge^{m+1} \pi_{p+1}^*(M_{p,B}) \otimes pr_1^*(L) \otimes \cdots \otimes pr_p^*(L) \otimes pr_{p+1}^*(L \otimes B^{-1}) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p} \Delta_{i,j}))$$

and the vanishing follows from Künneth on $C^{p+1} = C^p \times C$, using $H^1(L \otimes B^{-1}) = 0$ on the last component and the induction assumption on C^p for the tensor product of the remaining factors.

Case 2.2. $l > 0$: Diagonals involving the $(p + 1)$ -st component now appear with a positive multiplicity l . We wish to use the Leray spectral sequence for $pr_{p+1,*}$. We have to consider $H^1 \circ R^{k+l-1} pr_{p+1,*}$ and $H^0 \circ R^{k+l} pr_{p+1,*}$.

The fiber of $R^{k+l-1} pr_{p+1,*}$ of the sheaf in (7) over a fixed point $x \in C$ consists of

$$(8) \quad H^{k+l-1}(C^p, \wedge^{m+l+1} M_{p,B} \otimes pr_1^*(L + lx) \otimes \cdots \otimes pr_p^*(L + lx) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p} \Delta_{i,j})).$$

As the vanishings of $H^1 L$ and $H^1(B^{-1} \otimes L)$ imply the same for $L + lx$ in place of L , the cohomology group (8) vanishes by the induction assumption (note that $k + l - 1 \geq 1$).

Analogously, the same can be done for the fiber of $R^{k+l} pr_{p+1,*}$ of the sheaf in (7). \square

If the degree of B is large enough, then we can employ a suitable filtration of M_B and obtain a stronger vanishing result. Note that any line bundle of degree $\geq 2g + p$ is p -very ample.

Proposition 3.2. *If $\deg B \geq 2g + 2p + 1$ and $\deg L \geq 2g + 2p$, then the vanishing (4) holds, i.e.,*

$$H^1(C^{p+1}, M_B \otimes pr_1^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0.$$

Proof. Step 1. The proof of [9, IV 6.1] can be adapted to show that the general line bundle D of degree $g + 2p + 1$ is non-special and p -very ample. Here “general” means “after excluding a finite number of lower-dimensional subschemes” of the variety of line bundles of this degree.

Step 2. As $\deg(B \otimes D^{-1}) \geq (2g + 2p + 1) - (g + 2p + 1) = g$, there is a non-zero homomorphism $D \rightarrow B$ vanishing in a finite set of points z_i (possibly with higher multiplicities).

If all points z_i have multiplicity 1, we obtain an exact sequence

$$(9) \quad 0 \longrightarrow M_{p+1,D} \longrightarrow M_{p+1,B} \longrightarrow \oplus_i \mathcal{O}(-z_i, \dots, -z_i) \longrightarrow 0$$

and the required vanishing will follow from

$$(10) \quad H^1(C^{p+1}, M_{p+1,D} \otimes pr_1^* L \otimes \cdots \otimes pr_{p+1}^* L \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0.$$

and

$$(11) \quad H^1(C^{p+1}, L(-z_i) \otimes \cdots \otimes L(-z_i) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0$$

If there are points with higher multiplicity, then the sheaf on the right-hand side of (9) may not split into a direct sum, but will have a filtration whose graded pieces have the form (11).

Step 3. As $\deg(L \otimes D^{-1}) \geq 2g + 2p - (g + 2p + 1) = g - 1$, we have $H^1(L \otimes D^{-1}) = 0$ for general D , and the vanishing (10) follows from (3.1).

The vanishing (11) follows by induction (use (3.2) for $p - 1$) after applying $\pi_{p+1,*}$, given that $\deg L(-z_i) \geq 2g + 2p - 1 = 2g + 2(p - 1) + 1$. \square

A result for even lower degree of B is possible, and requires $\deg L$ to be higher by 1.

Proposition 3.3. *If $\deg B \geq 2g + p + 1$ and $\deg L \geq 2g + 2p + 1$, then the vanishing (4) holds, i.e.,*

$$H^1(C^{p+1}, M_B \otimes (\otimes_{i=1}^{p+1} (pr_i^*(L))) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0$$

Sketch of proof. Step 1. Successive applications of Leray's spectral sequence show that

$$\begin{aligned} & H^1(C^{p+1}, M_{p+1,B} \otimes pr_1^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) \\ & \cong H^1(C^{p+2}, pr_1^*(B) \otimes pr_2^*(L) \otimes \cdots \otimes pr_{p+2}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+2} \Delta_{i,j})) \\ & \cong H^1(C^{p+1}, M_{p+1,L} \otimes pr_1^*(B) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) \end{aligned}$$

By filtering $M_{p+1,L}$ instead of $M_{p+1,B}$ as in the proof of (3.2), we can reduce the question to the vanishing of

$$(12) \quad H^1(C^{p+1}, M_{p+1,D} \otimes pr_1^*(B) \otimes pr_2^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j}))$$

where D is a general line bundle of degree $g + 2p + 1$.

Step 2. The same steps as in the proof of (3.1) can be followed to show that

$$H^k(C^{p+1}, \wedge^m M_{p+1,D} \otimes pr_1^*(B) \otimes pr_2^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0$$

for all $k \geq 2$, $m > 0$, as long as $H^1 L = 0 = H^1(L \otimes D^{-1})$, no assumption on B .

Step 3. There exist line bundles $D_0, \dots, D_p = D$ on C with the following properties:

- (i) D_i is non-special of degree $g + 1 + 2i$ and i -very ample
- (ii) $H^0(C, D_{i+1} \otimes D_i^{-1})$ contains a nonzero section vanishing in two points x_i, y_i .

Step 4. The vanishing of (12) now follows from the case $m = 1$ of the next claim:

If D_p is general of degree $\deg(D_p) = g + 2p + 1$, $H^1 L = 0 = H^1(L \otimes D_p^{-1})$ and

$$\deg(B) \geq \deg(D_p) - p + m + g - 1,$$

then we have

$$H^1(C^{p+1}, \wedge^m M_{p+1,D_p} \otimes pr_1^*(B) \otimes pr_2^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0.$$

The induction uses a resolution step (as in 3.1), followed by a filtration step (as in 3.2). We leave the details to the reader. \square

Remark 3.4. Both (3.2) and (3.3) would follow from a more general vanishing result for

$$H^1(C^n, pr_1^*(L_1) \otimes \cdots \otimes pr_n^*(L_n) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq n} \Delta_{i,j})).$$

It seems likely that it is sufficient to assume that $\deg L_i \geq 2g - 3 + n + i$; under this condition, the vanishing is known for $n \leq 3$.

4. ADDITIONAL VANISHING RESULTS FOR THE CANONICAL BUNDLE

The filtration approach of 3.2 depends on finding a p -very ample line bundle D and a nonzero homomorphism $D \rightarrow B$ with the property that $h^0 D = h^0 B - (\deg B - \deg D)$ (equivalent to $h^1 D = h^1 B$). In geometric terms, the embedding of C by $H^0 D$ is the result of a sequence of inner projections of the embedding by $H^0 B$ from the points of a nonzero divisor in $H^0(B \otimes D^{-1})$.

If no such D exists, we obtain information on the geometry of the embedding by $H^0 B$. In the case $B = K$, this information can be interpreted in terms of the varieties of special divisors on C .

Proposition 4.1. *Suppose B is p -very ample, and there exists a p -very ample line bundle D of degree $\deg(B) - k$, and a non-zero homomorphism $D \rightarrow B$ which is an isomorphism on H^1 . If $\deg L \geq 2g + 2p$ and $\deg L \geq 2g - 1 + \deg B - 2k$, then the vanishing (4) holds, i.e.,*

$$H^1(C^{p+1}, M_B \otimes pr_1^*(L) \otimes \cdots \otimes pr_{p+1}^*(L) \otimes \mathcal{O}(-\sum_{1 \leq i < j \leq p+1} \Delta_{i,j})) = 0.$$

Proof. Let x be a point on C . The two conditions “ $B(-x)$ is p -very ample” and “ $B(-x) \rightarrow B$ is an isomorphism on H^1 ” are both open in C . Accordingly our assumptions still hold, if we replace D by $B \otimes \mathcal{O}(-E)$ for a general effective divisor E of degree $\deg(B) - \deg(D)$.

Going back to the proof of proposition (3.2), all that needs to be shown is that there exists an effective divisor E such that $H^1(L \otimes (B \otimes \mathcal{O}(-E))^{-1}) = 0$.

Now the variety of such effective divisors E has dimension k , whereas the variety of line bundles of degree $\deg(L \otimes B^{-1} \otimes \mathcal{O}(E))$ with nontrivial H^1 has dimension $2g - 2 - (\deg(L) - \deg(B) + k)$.

Therefore we are assured of a line bundle with vanishing H^1 as long as $k > 2g - 2 - (\deg(L) - \deg(B) + k)$, i.e., $\deg L > 2g - 2 + \deg(B) - 2k$. \square

Corollary 4.2. *Suppose there exists a p -very ample special line bundle D of degree $g + 2p$ with $\dim H^1 D = 1$. Then $K_{p,1}(C, K; L) = 0$ for any line bundle L with $\deg L \geq 2g + 4p + 1$.*

Proof. In order to apply 4.1, we need $\deg L \geq 2g - 1 + \deg K - 2(\deg K - \deg D) = 2g + 4p + 1$. The homomorphism $D \rightarrow K$ is due to $\text{Hom}(D, K) \cong H^0(K \otimes D^{-1}) \cong H^1(D)^\vee$. \square

Proposition 4.3. *Let C be a curve of genus g , $p \geq 1$, $1 \leq k \leq g - 2 - 2p$. The following conditions are equivalent:*

- (i) *There exists a p -very ample line bundle D of degree $2g - 2 - k$ with $\dim H^1 D = 1$.*
- (ii) *$\dim W_{p+j+2}^1 < j$ for all $0 \leq j \leq k - 1$.*

Proof. Either condition implies that K is p -very ample. Note also that either condition for some value of k implies the same condition for all lower values.

(i) implies (ii): By induction we can suppose that (ii) holds for all $0 \leq j \leq k - 2$. Now suppose $\dim W_{p+k+1}^1 \geq k - 1$, and let x_1, \dots, x_k be an arbitrary sequence of points (possibly with repetitions). Then the internal projection of C from the points x_1, \dots, x_k into \mathbf{P}^{g-k} is not p -very ample: One sees inductively (for each $l \leq k - 1$) that the subvariety of W_{p+k+1}^1 that consists of all pencils with x_1, \dots, x_l as base locus, has dimension $\geq k - 1 - l$. Considering $l = k - 1$, there will be a g_{p+k+1}^1 with base locus x_1, \dots, x_{k-1} , hence the projection from x_1, \dots, x_k will have a $(p+1)$ -secant $(p-1)$ -plane and $K_C \otimes \mathcal{O}_C(-x_1 \dots - x_k)$ will not be p -very ample.

(ii) implies (i): Suppose $\dim W_{p+j+2}^1 < j$ for all $0 \leq j \leq k - 1$, and assume we have determined points $x_1, \dots, x_l \in C$ so that each of the subvarieties of W_{p+j+2}^1 consisting of pencils with base locus x_1, \dots, x_l has dimension $< j - l$. Given an irreducible component of such a subvariety of W_{p+j+2}^1 , the general point of C will not be a base point of each of the pencils in this component; i.e., for general x , the subvariety of pencils with base locus x_1, \dots, x_l, x will have dimension $< j - l - 1$. As each such subvariety of W_{p+j+2}^1 has only finitely many components, and there are only finitely many values of j to consider, we conclude that there is a point $x_{l+1} \in C$ so that each of the subvarieties of W_{p+j+2}^1 consisting of pencils with base locus x_1, \dots, x_{l+1} has dimension $< j - l - 1$.

Finally, for $l = k$ we have points x_1, \dots, x_k with the property that there is no pencil in W_{p+k+2}^1 with these points as base locus, i.e., $K_C \otimes \mathcal{O}_C(-x_1 \dots - x_k)$ is p -very ample. \square

Remark 4.4. We list some of the consequences:

1. Suppose $\text{gon}(C) \geq p + 3$, i.e., $W_{p+2}^1 = \emptyset$ and K_C is $(p + 1)$ -very ample. Then $K_{p,1}(C, K; L) = 0$ for $\deg L \geq 4g - 5$. Similarly, if $\text{gon}(C) \geq p + 3$ and $\dim W_{p+3}^1 \leq 0$ (i.e., W_{p+3}^1 is empty or zero-dimensional), then $K_{p,1}(C, K; L)$ vanishes for $\deg L \geq 4g - 7$.
2. Suppose C fulfils the conditions of (4.3(ii)) for $k = g - 2 - 2p$, and let $0 \leq j \leq p$. Then $K_{j,1}(C, K; L) = 0$ for $\deg(L) \geq 2g + 4j + 1$.

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